1. *Answer true or false. No marks will be awarded in the absence of proper justi cation.*

   (a) Let A be a $n \times n$ matrix such that $A^2 = A$ and rank$(A) = n$. Then $A = I$.

   Solution: True. Since rank$(A) = n$, A is invertible, then $A^2 = A$ implies that $A^{-1}.A^2 = A^{-1}.A$, which in turn, implies that $A = I$. □

   (b) If row space of a $n \times n$ matrix A equals its column space, then $A = A^t$.

   Solution: False. Let us consider the matrix

   $$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

   Then row space of A equals its column space, but $A \neq A^t$. □

   (c) Only possible eigenvalues of a 3×3 symmetric orthogonal matrix are 1 and -1.

   Solution: True. Since $AA^t = I$ and $A = A^t$, it implies that $A^2 = I$. If λ is an eigenvalue of A then $\lambda^2 = 1$. Hence possible values of λ are 1 and -1. □

   (d) If A is a complex $n \times n$ matrix such that X^*AX is real for all $X \in C^n$, then A is Hermitian.

   Solution: True. For any $X \in C^n$, we have

   $$\langle AX, X\rangle = X^*AX \text{ is a real number.}$$

   Let us consider standard notation for unit vectors e_j and e_k, then we have

   $$\langle A(e_j + e_k), e_j + e_k\rangle = a_{jj} + a_{kk} + a_{jk} + a_{kj}$$

   i.e., $a_{jk} + a_{kj}$ is real since diagonal entries are real ($a_{ii} = \langle Ae_i, e_i\rangle$). In turn, it implies that $Im(a_{jk}) = -Im(a_{kj})$. Similarly,

   $$\langle A(i.e_j + e_k), i.e_j + e_k\rangle = a_{jj} + a_{kk} - i.a_{jk} + i.a_{kj}$$

   i.e., $i.(a_{kj} - a_{jk})$ is real. Hence, $Re(a_{kj}) = Re(a_{jk})$.
   Thus, for any $1 \leq j, k \leq n$, we get $Im(a_{jk}) = -Im(a_{kj})$ and $Re(a_{kj}) = Re(a_{jk})$, which implies $A = A^*$. □

(e) Eigenvalues of a real symmetric matrix are real.

Solution: True. Let A be a real symmetric matrix, λ be an eigenvalue of A, and x be the corresponding eigenvector. Then

$$\begin{aligned} Ax &= \lambda x \\ x^* A^* &= \bar{\lambda} x^* \\ x^* Ax &= \bar{\lambda} x^* x \\ \lambda x^* x &= \bar{\lambda} x^* x \end{aligned}$$

Since $x \neq 0$, $x^* x \neq 0$ (Here, $-^*$ denotes conjugate transpose). Hence, $\lambda = \bar{\lambda}$, i.e. λ is a real number. $\square$

2. Let A; B be $m \times n$ matrices over a field F. Prove that rank$(A + B) \leq$ rank(A)+ rank(B).

Solution: Let $\{a_1, \cdots, a_n\}$ be the set of columns in A and $\{b_1, \cdots, b_n\}$ be the set of columns in B. Then $a_1 + b_1, \cdots, a_n + b_n$ denotes the set of columns of $A + B$.

Since Span$\{a_i, b_j\}$ contains the Span$\{a_i + b_i\}$ for $1 \leq i, j \leq n$, it follows that the dimension of column space of $A + B$ is less than or equal to the sum of dimensions of column spaces of A and B.

We know that rank of a matrix is same as the dimension of the column space (or row space) of the matrix. Hence,

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

$\square$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

(a) Find the characteristic polynomial of A.

Solution: Let us consider the characteristic equation $det(A - \lambda I) = 0$, which gives the following expression.

$$\begin{aligned} \{-\lambda(2 - \lambda) + 1\}\{(1 - \lambda)(-1 - \lambda) + 1\} &= 0 \\ \lambda^2 (1 - \lambda)^2 &= 0. \end{aligned}$$

The characteristic polynomial of A is $x^2(1 - x)^2$.

(b) Find the minimal polynomial of A.

Solution: Let us observe that by matrix multiplication it follows that

$$A(A-I) \neq 0$$
$$A^2(A-I) \neq 0$$
$$A(A-I)^2 \neq 0.$$

Therefore, the minimal polynomial of A is same as the characteristic polynomial $x^2(1-x)^2$. We used the fact that minimal polynomial divides the characteristic polynomial and the matrix A satisfies the minimal polynomial.

(c) Is A diagonalizable over $\mathbb{C}$? Give reasons.

Solution: The matrix A is diagonalisable over $\mathbb{C}$ if and only if all the roots of the minimal polynomial of A are of algebraic multiplicity 1, i.e. minimal polynomial is a product of distinct linear factors over $\mathbb{C}$. Since, for the matrix A the minimal polynomial does not satisfy the above mentioned necessary condition. So, A is not diagonalizable.

4. Let V be the space of all real polynomials of degree at most 3.

(a) Prove that

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \ \ \forall f, g \in V$$

defines a positive definite symmetric bilinear form on V.

Solution: By the definition of $\langle , \rangle : V \times V \to V$ it is straightforward to check that

$$\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$$
$$\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$$
$$\langle \lambda.f, g\rangle = \lambda.\langle f, g\rangle = \langle f, \lambda.g\rangle$$
$$\langle f, g\rangle = \langle g, f\rangle$$

Thus, $\langle , \rangle$ is a symmetric bilinear form on V. Next, we have to show that for a non-zero $f \in V$ we get $\langle f, f\rangle > 0$. Since $f \neq 0$, $f^2 \neq 0$ and $f^2(x) \geq 0$ for $x \in [0,1]$ which implies that $\int_0^1 f^2(x)dx > 0$. (Note that f^2 is a polynomial and it can be zero at only finitely many points in $[0,1]$. Thus, the area under the curve f^2 in $[0,1]$ is positive.)

(b) Find the orthogonal complement of the subspace of scalar polynomials.

Solution: Let S be the subspace of scalar polynomial and $S^{\perp}$ be the orthogonal complement of S. If $f \in S^{\perp}$, then

$$\langle f, c\rangle = \int_0^1 c.f(x)dx = 0.$$

Let $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $\langle f, c\rangle = 0$ implies that

$$c = 0 \text{ or } \frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0.$$

Thus, $S^{\perp} = \{f = \sum_{i=0}^n a_i.x^i \in V | \frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0\}$

(c) Apply Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$ to find an orthonormal basis of $(V, \langle, \rangle)$.

Solution: Let us start with the basis $\{1, x, x^2, x^3\}$ and apply Gram-Schmidt process to find an orthonormal basis. Let $v_1 = 1$, then v_2 is given by

$$v_2 = x - \langle x, 1/2\rangle = x - 1/2.$$

Now, $||v_2||^2 = \int_0^1 (t - 1/2)^2 dt = 1/12$.
Next,

$$v_3 = x^2 - \frac{\langle x^2, v_1\rangle}{||v_1||^2}.v_1 - \frac{\langle x^2, v_2\rangle}{||v_2||^2}.v_2$$

i.e., $v_3 = x^2 - x + 1/6$. Also,

$$||v_3||^2 = 1/36.$$

Then,

$$v_4 = x^3 - \frac{\langle x^3, v_1\rangle}{||v_1||^2}.v_1 - \frac{\langle x^3, v_2\rangle}{||v_2||^2}.v_2 - \frac{\langle x^3, v_3\rangle}{||v_3||^2}.v_3$$

i.e., $v_4 = x^3 - \frac{3}{10}(x^2 - x + \frac{1}{6}) - \frac{9}{10}(x - \frac{1}{2}) - \frac{1}{4}$.
Hence, an orthonormal basis of V is given by

$$\{1, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||}, \frac{v_4}{||v_4||}\}.$$

5. (a) Prove that a complex matrix M is normal if and only if there is a unitary matrix P such that P^*MP is diagonal.

Solution: Let us first assume that there exists a unitary matrix P such that P^*MP is diagonal. Next, observe that diagonal matrices commute and therefore, diagonal matrices are normal. If $P^*MP = D$ for a diagonal matrix D, then M is normal.

Conversely, let M be normal. By the Schur decomposition, the matrix M can be written as $M = P^*TP$, where P is unitary matrix and T is an upper-triangular matrix. Since A is normal, it follows that

$$TT^* = T^*T.$$

Therefore, T must be diagonal since a normal upper triangular matrix is diagonal. □

(b) Hence show that every conjugacy class in the unitary group $U_n(C) = \{P \in \mathbb{C}_{n\times n} : P^*P = I\}$ contains a diagonal matrix.

Solution: Let us consider conjugacy class of an element $Q \in U_n(C)$. Since $Q^*Q = I = QQ^*$, it follows that Q is a normal matrix. Thus, there exists a matrix $P \in U_n(C)$ such that $P^*QP = P^{-1}QP = D$, where D is a diagonal matrix. Hence, every conjugacy class in $U_n(C)$ contains a diagonal matrix.

6. Let

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

(a) Find the eigenvalues of A and corresponding eigenvector.

Solution: The characteristic polynomial of A is $x^2 - 5x + 4$. Hence, the eigenvalues are $1, 4$. The corresponding eigenvectors are solutions of the equation:

$$(A - \lambda.I)X = 0.$$

Note that $X \in \mathbb{C}^2$. It follows that the eigenvector corresponding to eigenvalue 1 is:

$$X_1 = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

and the eigenvector corresponding to eigenvalue 4 is:

$$X_2 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}.$$

□

(b) Find a unitary matrix P such that P^*AP is a diagonal matrix.

Solution: Note that $||X_1|| = ||X_2|| = \sqrt{3}$. Since A is a Hermitian matrix, let us define a matrix P with column vectors $\frac{X_1}{||X_1||}$ and $\frac{X_2}{||X_2||}$.

$$P = \frac{1}{\sqrt{3}} \begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}$$

Note that $PP^* = I$, i.e. P is a unitary matrix. Moreover,

$$P^*AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

□